Weierstrass Theorem, Contraction Mapping Theorem

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Compact Sets

Let $\mathbb{U} = \{U_i, i \in I\}$ denote a collection of sets. For example, $\mathbb{U} = \{U_1, U_2, \cdots\}$ if $I = \mathbb{N}$.

Definition (Cover and Open Cover)

- Cover: $\mathbb{U}$ is a cover of the set A if $A \subset \cup_{i \in I} U_i$.
- $\{U_i, i \in S\}$ is a *subcover* if $A \subset \cup_{i \in S} U_i$ and $S \subset I$. The subcover is finite if S is a finite set.
- Open cover: $A \subset \cup_{i \in I} U_i$ and U_i is open $\forall i \in I$.

Definition (Compact Set)

A set A is compact if *every* open cover of A has a finite subcover.

Compact Sets

An example for an open cover of A is $\mathbb{B} = \{B_{\epsilon(x)}(x), x \in A\}$ where $B_\epsilon(x)$ is an open ball centered at x with radius ϵ.

Exercise

Use the definitions to show the following:

- $\mathbb{R}$ *is not compact.*
- $(0, 1)$ *is not compact.*
- $[0, 1]$ *is compact.*

Compact Sets

We have seen previously that a set A is closed if and only if every convergent sequence $\{x_n\}$ contained in A converges to a point $x \in A$. Similar results hold with compact sets.

Theorem

In a metric space (X, d), a set $A \subset X$ is compact if and only if every sequence $\{x_n\}$ with $x_n \in A$ has a convergent subsequence with limit in A.

We will only prove the 'only if' part in class.

Compact Sets in Euclidean Spaces

Theorem (Heine-Boreal)

Any closed and bounded set of real numbers is compact.

Proof.

To prove this result, we first show that any closed subset of a compact set is also compact. Then apply the Bolzano-Weierstrass Theorem. □

The Heine-Borel theorem also holds for finite dimensional Euclidean spaces. Any closed and bounded subset of $\mathbb{R}^m$ is compact.

The converse of the previous theorem also holds for sets in a metric space.

Theorem

A compact set in a metric space is closed.

Proof.

Use the sequence characterizations of a compact and a closed set. □

A compact set in a metric space is also bounded.

Cantor Set

The Cantor set is constructed by iteratively deleting the open middle third from a set of line segments. Let's start from $[0, 1]$.

$$
\begin{aligned}
C_0 &= [0, 1] \\
C_1 &= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\
C_2 &= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \\
&\dots
\end{aligned}
$$

The Cantor set is $C = \cap_{k=0}^{\infty} C_k$ which contains all points in $[0, 1]$ not deleted in any step above.

Exercise

Show that the Cantor set is compact.

The following theorem shows that the continuous image of a compact set is also compact.

Theorem

(X, d) and (Y, ρ) are metric spaces and $f : X \to Y$ is a continuous function. If $C \subset X$ is compact, $f(C)$ is also compact.

Proof.

The theorem can be proved by showing that any sequence in $f(C)$ has a convergent subsequence that converges to a point in $f(C)$. Alternatively, we can use the definition of a compact set and a previous result that if f is continuous and $U \subset Y$ is open, $f^{-1}(U)$ is also open. □

Exercise

Show that for continuous function $f : \mathbb{R} \to \mathbb{R}$*, if* $C \subset \mathbb{R}$ *is a compact interval, so is* $f(C)$*.*

Proof.

An intermediate step is to show that if f is continuous on $[a, b]$, for any c between $f(a)$ and $f(b)$, there is some $x \in [a, b]$ that $c = f(x)$. □

Theorem (The Weierstrass Theorem)

If $f : X \to \mathbb{R}$ is a continuous function, and the set $C \subset X$ is compact, there exists $x_M, x_m \in C$ such that $f(x_M) = \sup_{x \in C} f(x)$ and $f(x_m) = \inf_{x \in C} f(x)$.

Proof.

Because $f(C)$ is compact in $\mathbb{R}$, it is closed and bounded. The supremum and infimum exist because $f(C)$ is bounded, and they are elements of $f(C)$ because $f(C)$ is closed. □

Example: The Solow Model

The economy has k_t capital stock that can be used to produce $f(k_t) = k_t^\alpha$ output with $0 < \alpha < 1$. A fixed share of the output, s, will be saved and the rest will be consumed. Capital depreciates at rate δ.

$$c_t = (1-s)f(k_t),$$
$$k_{t+1} = (1-\delta)k_t + sf(k_t)$$

In the steady state, $k_t = k_{t+1} = k^*$, which solves

$$k^* = (1-\delta)k^* + s(k^*)^\alpha.$$

Example

Show that for any k_0, the sequences $\{k_t\}$ converges monotonically to k^*.

Definition (Contraction)

Let (X, d) be a metric space and T a function from $X \to X$. T is a contraction of modulus β if $\forall x, y \in X$, $d(Tx, Ty) \leq \beta d(x, y)$ for $0 < \beta < 1$.

- A *fixed point* x of T satisfies $Tx = x$.
- A metric space (X, d) is *complete* if it contains the limit of every Cauchy sequence $\{x_n\}$ with $x_n \in X$, that is $\lim x_n \in X$.
- Composite functions: $T^2(x) = T(T(x))$, etc.

Theorem (Contraction Mapping Theorem)

$T : X \to X$ is a contraction with modulus $0 < \beta < 1$ on a complete metric space (X, d).

1. *The fixed point x^* of T exists and is unique.*
2. *Define the sequence $\{x_n(x_0)\}$ with $x_n = T^n(x_0)$. $\lim_{n\to\infty} x_n(x_0) = x^*$ for any x_0.*

(2) provides a method of successive approximations, as $d(x_{n+1}, x^*) = d(Tx_n, Tx^*) \leq \beta d(x_n, x^*)$. The solution is unique and is not dependent on the initial value.

Exercise

$T : X \to X$ is a function on a complete metric space (X, d). Show that if T^n is a contraction, T has a unique fixed point.

Let function $f : \mathbb{R}^n \to \mathbb{R}$, and $B(\mathbb{R}^n, \mathbb{R})$ be the set of all such bounded functions. The metric is the *sup norm*, $||f||_s = \sup\{|f(x)|, x \in \mathbb{R}^n\}$.

Theorem (Blackwell's Sufficient Conditions for a Contraction)

Let $T : B(\mathbb{R}^n, \mathbb{R}) \to B(\mathbb{R}^n, \mathbb{R})$. It is a contraction if it satisfies the following two conditions:

- *Monotonicity:* $\forall f, g \in B(\mathbb{R}^n, \mathbb{R}), f \leq g \Rightarrow Tf \leq Tg$.
- *Discounting:* $\exists \beta \in (0, 1)$, *such that* $\forall f \in B(\mathbb{R}^n, \mathbb{R}), x \in \mathbb{R}^n, \alpha > 0$, $T(f(x) + \alpha) \leq T(f(x)) + \beta\alpha$.

Example: Optimal Growth Model

The economy now has k_t capital, which can be used to produce $f(k_t)$ output. Some of the output can be consumed (c_t) which provides utility, and some can be used to purchase capital that can be used in the next year k_{t+1}. The problem is to choose a sequence of $\{c_t\}_{t=0}^{\infty}$ and $\{k_{t+1}\}_{t=0}^{\infty}$, such that

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to the constraint that $c_t + k_{t+1} \leq f(k_t)$, $c_t \geq 0$ and $k_{t+1} \geq 0$.

Example: Optimal Growth Model

Define the value function $v(k_0) = \sum_{t=0}^{\infty} \beta^t U(c_t)$ where the optimal sequences of $\{c_t\}_{t=0}^{\infty}$ and $\{k_{t+1}\}_{t=0}^{\infty}$ are chosen. The value function is a solution to the following problem (by Bellman's Principal of Optimality)

$$v(k) = \max_{c,k'} \left(U(c) + \beta v(k')\right)$$

subject to $c + k' \leq f(k)$ and $c, k' \geq 0$.
The recursive formulation of optimization problems appears in many other contexts. For example

- Bus engine replacement, Rust (1994)
- Retirement from the labor force, Rust and Phelan (1997)

Example: Optimal Growth Model

If $U(c)$ is strictly increasing in c, the above problem can be written as

$$v(k) = \max_{k'} \left(U(f(k) - k') + \beta v(k') \right)$$

subject to $k' \geq 0$.
Define a function $T : B(\mathbb{R}, \mathbb{R}) \to B(\mathbb{R}, \mathbb{R})$,
$Tv(k) = \max_{k'} \left(U(f(k) - k') + \beta v(k') \right)$. The solution v^* is a fixed point of T.

Exercise

Use Blackwell's Sufficient Conditions to show that T is a contraction.