Differential Calculus

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Derivatives

We first consider a univariate function $g : \mathbb{R} \to \mathbb{R}$.

Definition (Derivatives)

The derivative of g at x, $g'(x)$, if it exists, is

$$\lim_{y \to x} \frac{g(y) - g(x)}{y - x} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h},$$

and g is differentiable at x. g is differentiable if it is differentiable on every point in its domain.

If $g'(x)$ exists, the limit of the quotient $\frac{g(x+h)-g(x)}{h}$ from the left ($h \to 0^-$) must equal to the limit from the right ($h \to 0^+$).

The derivative at x can be seen as the limit of the secants between $(x, g(x))$ and $(y, g(y))$ as y approaches x.
$g'(x)$ describes the slope at x, or the rate of change of $g(x)$ at x.
Higher order derivatives can be defined similarly. g'' describes the curvature of g.

Theorem

If g is differentiable at x, it is also continuous at x.

Proof.

Use the result that if $\{x_n\} \to x$ and $\{y_n\} \to y$, then $\{x_n y_n\} \to xy$. □

If the n-th order derivative, g^n exists and is also continuous, g is called n-th order continuously differentiable.

Derivatives

Let f and g be functions $\mathbb{R} \to \mathbb{R}$ and assume that they are differentiable at x_0. The following are some rules for differentiations.

- $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
- (product rule) Define $p(x) = f(x)g(x)$. $p'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$.
- Define $p(x) = f(x)/g(x)$ and $g(x_0) \neq 0$. $p'(x_0) = \frac{g(x_0)f'(x_0) - g'(x_0)f(x_0)}{g^2(x_0)}$.
- (chain rule) Define $p(x) = f(g(x))$. $p'(x_0) = f'(g(x_0))g'(x_0)$.

Definition (Local Maximizer)

x_0 is a local maximizer of f if for some $\delta > 0$, $f(x_0) \geq f(x)$ for all $x \in B_\delta(x_0)$. Local minimizer can be defined similarly.

The following theorem provides a necessary condition for x_0 as a local maximizer for an interior point.

Theorem (first-order condition)

$f : (a, b) \to \mathbb{R}$ is differentiable. If x_0 is a local maximizer, then $f'(x_0) = 0$.

Proof.

Consider the limit as $h \to 0^-$ and $h \to 0^+$. The derivative at x_0 exists only if the left and right limits are equal. □

Exercise (sufficient condition for a local maxmizer)

$f : (a, b) \to \mathbb{R}$ is twice continuously differentiable and $x_0 \in (a, b)$. Show that if $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a local maximizer of f.

Theorem (Rolle's Theorem)

$f : [a, b] \to \mathbb{R}$ *is differentiable. If* $f(a) = f(b) = 0$*, then there exists some* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

Proof.

Use the Weierstrass Theorem. $\square$

Theorem (Mean-Value Theorem)

$f : \mathbb{R} \to \mathbb{R}$ is differentiable. There exists a point $c \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(c)$.

Proof.

Subtract the secant line between $(a, f(a))$ and $(b, f(b))$ from $f(x)$, and apply Rolle's theorem. □

We state the following results without proof.

Theorem (Taylor's Formula)

$f : \mathbb{R} \to \mathbb{R}$ is n-th order differentiable on (a, b). For $x, x + h \in (a, b)$, there exists a point $c \in (x, x + h)$,

$$f(x+h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} h^k + \frac{f^{(n)}(c)}{n!} h^n.$$

Theorem (L'Hôpital's Rule)

$f, g : \mathbb{R} \to \mathbb{R}$ are differentiable on an open interval containing point a, $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Derivatives of a Multivariate Function

Consider a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$. Let u be a vector in $\mathbb{R}^n$. For some point $x^0 \in \mathbb{R}^n$, $x^0 + \alpha u$, $\alpha \in \mathbb{R}$ describes a straight line in $\mathbb{R}^n$ through x^0 in the direction of u. The derivative of f at x^0 if we approach it along this line is the directional derivative of f in the direction of u at x^0.

Definition (Directional Derivative)

$$Df(x^0, u) = \lim_{\alpha \to 0} \frac{f(x^0 + \alpha u) - f(x^0)}{\alpha},$$

where $\alpha \in \mathbb{R}$ and $||u|| = 1$.

Definition (Partial Derivative)

Let $e^i \in \mathbb{R}^n$ where all components are zero except for the i-th element which is 1. The partial derivative of f with respect to its i-th argument at x^0 is $Df(x^0, e^i)$, which can be denoted as $D_{x_i} f(x^0)$, $f_{x_i}(x^0)$, $f_i(x^0)$, or $\frac{\partial f(x^0)}{\partial x_i}$.

Definition (Gradient Vector)

$$\nabla f(x) = (f_1(x), \cdots, f_n(x)),$$

an $1 \times n$ vector.

Exercise

Let $f(x_1, x_2) = x_1 x_2$, $u = (\frac{3}{5}, \frac{4}{5})$ *and* $x^0 = (1, 2)$. *Calculate* $Df(x^0, u)$ *according to the definition on the previous slide, and show that it equals* $\nabla f(x^0)u$.

Exercise

Let

$$f(x_1, x_2) = \begin{cases} \frac{xy^2}{x^2+y^4} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}$$

and $x^0 = (0, 0)$. *Calculate* $Df(x^0, u)$, *and show that it does not equal* $\nabla f(x^0)u$.

The previous exercise shows that the existence of directional derivatives is not sufficient for continuity. Another example is

$$f(x, y) = \begin{cases} 1 & \text{if } y = x^2 \text{ and } x \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

The directional derivative of f at $(0, 0)$ along any direction exists and is 0, but f is not continuous at $(0, 0)$.

Differentiability of a Multivariate Function

For $f : \mathbb{R} \to \mathbb{R}$, from the Taylor formula, for some $c \in (x, x+h)$,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(c)}{2}h^2.$$

The linear terms $f(x) + f'(x)h$ approximates $f(x+h)$, as the error term $E(h) = \frac{f''(c)}{2}h^2$ quickly vanishes as $h \to 0$, $\lim_{h\to 0} \frac{E(h)}{h} = 0$. For a differentiable function, linear approximation can approximate the function value well locally.

Definition (Differentiability)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and X an open set in $\mathbb{R}^n$. f is differentiable at $x \in X$ if there exists a $m \times n$ matrix A_x, such that

$$\lim_{||h||\to 0} \frac{||f(x+h) - f(x) - A_x h||}{||h||} = 0,$$

h is a vector in $\mathbb{R}^n$ and $||\cdot||$ is the Euclidean norm, $||h|| = \sqrt{h \cdot h}$. f is differentiable if it is differentiable at every point in its domain.

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in X \subset \mathbb{R}^n$, the derivative is the $m \times n$ matrix of first partial derivatives of f,

$$A_x = Df(x) = \begin{pmatrix} Df^1(x) \\ \cdots \\ Df^m(x) \end{pmatrix} = \begin{pmatrix} \nabla f^1(x) \\ \cdots \\ \nabla f^m(x) \end{pmatrix} = \begin{pmatrix} f^1_{x_1}(x) & \cdots & f^1_{x_n}(x) \\ \cdots & \cdots & \cdots \\ f^m_{x_1}(x) & \cdots & f^m_{x_n}(x) \end{pmatrix}.$$

The matrix $Df(x)$ is called the Jacobian of f at x.

Proof.

Consider the ith component of f, $f^i : \mathbb{R}^n \to \mathbb{R}$ and show that f is differentiable if and only if each f^i is differentiable. To show that $Df^m(x) = \nabla f^m(x)$, use e^i. □

The previous theorem shows that if f is differentiable at x, its directional derivative $Df(x, h) = Df(x)h$.

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in X \subset \mathbb{R}^n$, it is continuous at x.

Proof.

$||f(x+h) - f(x)|| \leq ||Df(x)h|| + ||f(x+h) - f(x) - Df(x)h||.$ □

The proof of the following results can be found in the textbook.

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$. If all the partial derivatives of f exist and are continuous at an interior point x, then f is differentiable.

Theorem (Chain Rule for Multivariate Composite Functions)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^p$ and $F(x) = g(f(x))$, $DF(x) = Dg(y)Df(x)$ with $y = f(x)$.

Theorem (Mean Value Theorem for Multivariate Functions)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable on an open set $X \subset \mathbb{R}^n$. Suppose that $x, y \in X$ and the line segment between x and y is also in X. For each $a \in \mathbb{R}^m$, there exists $\lambda \in [0, 1]$, such that

$$a' \left(f(y) - f(x)\right) = a' \left(Df(\lambda x + (1 - \lambda)y)(y - x)\right).$$

Taylor's Formula for Multivariate Functions

Theorem

$f : \mathbb{R}^n \to \mathbb{R}$ is second order differentiable. For some $\lambda \in (0, 1)$,

$$f(x+h) = f(x) + Df(x)h + \frac{1}{2}h'D^2 f(x+\lambda h)h.$$

Proof.

Define a univariate function $g(\alpha) = f(x + \alpha h)$ and apply the univariate Taylor's formula. □

The formula provides a first order approximation for a multivariate function. The remainder term is quadratic in h.

Continuous Differentiability

Definition (Continuously Differentiable Function)

If function $f : \mathbb{R}^n \to \mathbb{R}^m$ has continuous partial derivatives on some open set $X \subset \mathbb{R}^n$, it is continuously differentiable in X. All such functions are denoted C^1 functions.

Similarly, a C^0 function is a continuous function. A C^k function for some $k \geq 1$ has continuous k-th order partial derivatives.

Implicit Function

The graph of x and y that satisfy $u(x, y) = c$ describes an indifference curve. We may be interested in knowing how much the consumption of y needs to change with some change in x in order for one to be indifferent. If we can write $y = f(x)$, $u(x, f(x)) = c$ and take derivative with respect to x,

$$u_x(x, y) + u_y(x, y)f'(x) = 0$$

The marginal rate of substitution, $f'(x)$ is

$$f'(x) = -\frac{u_x(x, y)}{u_y(x, y)}.$$

For a general function $f(x, y)$, we would like to know when y can be expressed as a function of x.

Implicit Function Theorem

Theorem (Implicit Function Theorem)

$f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Suppose $f(x_0, y_0) = c$. If $f_y(x_0, y_0) \neq 0$, in some neighborhood of (x_0, y_0), $y = g(x)$ and g is continuously differentiable, that satisfies $f(x, g(x)) = c$.

Example

Let $f(x, y) = y - g(x)$. Consider $f(x, y) = 0$. The inverse function of g, $x = g^{-1}(y)$ satisfies $f(g(y), y)$. From the implicit function theorem, g^{-1} exists if $f_x(x, y) \neq 0$.

Homogeneous Functions

A set $X \subset \mathbb{R}^n$ is a cone if for any $x \in X$, $\lambda x \in X$ for any $\lambda > 0$.

Definition (Homogeneous Function)

f is homogenous of degree k in a cone X if for all $\lambda > 0$ and $x \in X$, $f(\lambda x) = \lambda^k f(x)$.

Examples of homogeneous functions

- demand function $x(p, y)$
- $f(x_1, x_2) = \frac{x_1}{x_2}$
- Cobb-Douglas function $f(x) = A \prod_{i=1}^{n} x_i^{\alpha_i}$ is homogeneous of degree $\sum_{i=1}^{n} \alpha_i$.

Euler's Theorem

Theorem (Euler's Theorem)

$f : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives on an open cone X. It if homogeneous of degree k in X if and only if $\forall x \in X$,

$$\sum_{i=1}^{n} f_i(x)x_i = kf(x).$$

Example

If the production function $f(x_1, x_2)$ is homogeneous of degree 1, and the price of each input relative to the output equals its marginal product, Euler's theorem implies that the profit is zero.

Exercise

Use Euler's theorem to show that $f(x) = A\prod_{i=1}^{n} x_i^{\alpha_i}$ is homogeneous of degree $\sum_{i=1}^{n} \alpha_i$.

Exercise

Show that if f is homogeneous of degree k, then f_i is homogeneous of degree $k - 1$.

- If utility function $u(x_1, x_2)$ is homogenous, the indifference curves are radial expansions of each other. The marginal rate of substitution between x_1 and x_2, or the slope of indifference curves, is constant along each straight line from the origin.
- The same geometric properties also hold for a *homothetic* function which is an increasing transformation of a homogeneous function.