Static Models and Comparative Statics

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Many economic systems can be described in general terms by

$$F(x; \alpha)$$

where x is a vector of *endogenous variables* and α is a vector of *parameters* or exogenous variables that describe the economic environment. If we set $F(x; \alpha) = 0$, the x that satisfy the equation is said to be in equilibrium. The model is *static* as we are not describing how one equilibrium is reached (time index t does not appear). We are interested in the following,

- Given α, the set of x that satisfy the equilibrium conditions.
- If we start from one equilibrium $F(x_0; \alpha_0) = 0$, and α_0 is changed to α_1, how will the new equilibrium x_1 be like?

Linear Systems

One example is the Keynesian national income model,

$$Y = C + I_0 + G_0,$$
$$C = a + bY$$

where $a > 0$ and $0 < b < 1$. Assuming that investment (I_0) and government expenditure (G_0) are exogenously given, the above system describes the relation between endogenous variables national income Y and consumption C.

$$\begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \end{pmatrix} = \begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix}.$$

The above system has a unique solution.

Solutions of Linear Systems

In general terms, we are interested in the solutions for

$$Ax = y,$$

where y is $m \times 1$, A is $m \times n$ and x is $n \times 1$. The following are possible scenarios.

- The system has no solution: y is not in the column space of A.
- The system has solutions: $\text{rank}A = \text{rank}\begin{pmatrix} A & y \end{pmatrix}$.
  - Multiple solutions. If $\ker A \neq \{0_n\}$, let $x^H \in \ker A$. If x^P is a solution of $Ax = y$, $x^P + x^H$ is also a solution of $Ax = y$.
  - Unique solution. This is the case when $\ker A = \{0_n\}$, which implies that $\text{rank}A = n$ and A must be a square matrix.
    - When the solution is unique, we can write $x^* = A^{-1}y$.
    - Cramer's rule:
      $$x_i^* = \frac{\det A_i}{\det A},$$
      where we replace the i-th column of A by y in A_i.

Comparative static analysis is easy when the endogenous variable can be written as an explicit function of the exogenous variables. Suppose that $x^* = A^{-1}y$. If A^{-1} and y are functions of scalar parameter α,

$$\begin{aligned}\frac{\partial x^*(\alpha)}{\partial \alpha} &= A^{-1}\frac{\partial y}{\partial \alpha} + \frac{\partial A^{-1}}{\partial \alpha}y \\ &= A^{-1}\frac{\partial y}{\partial \alpha} - A^{-1}\frac{\partial A}{\partial \alpha}A^{-1}y,\end{aligned}$$

which gives us how x^* will change for small changes in the parameter α.

Let us now consider more general models.

Example (Market Equilibrium)

The quantity demanded and supplied are given by functions

$$Q^d = D(P)$$
$$Q^s = S(P, \theta)$$

where θ is a parameter. Market equilibrium requires that $Q^d = Q^s$, which gives

$$D(P) = S(P, \theta). \tag{1}$$

Given θ, if there is a unique P^* such that $D(P^*) = S(P^*, \theta)$, then the (1) implicitly defines a function $P(\theta)$ and we can answer questions like how will changes in the parameter affect the equilibrium price.

Define $F(P,\theta) = D(P) - S(P,\theta)$. The market equilibrium in the previous example can be represented by

$$F(P,\theta) = 0.$$

In general, define a function $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x,\alpha)$. Fix the value of α, we have a univariate function $f_{\alpha^0}(x) = F(x,\alpha^0)$. Let

$$S(\alpha) = \{x \in X, f_\alpha(x) = 0\}.$$

We are interested in the following questions.

- When is $S(\alpha)$ a function?
- The shape of $S(\alpha)$, i.e. $S'(\alpha)$.

Implicit Function Theorem

Suppose we find one equilibrium, $F(x^0, \alpha^0) = 0$ and consider α that is close to α^0 ($\alpha \in B_\eta(\alpha^0)$). For $S(\alpha)$ to be a function on $B_\eta(\alpha^0)$, there must be a unique x such that $f_\alpha(x) = 0$. Intuitively, a necessary condition is that whenever we change the value of x, the value of $F(x, \alpha)$ must change. Otherwise the same α can satisfy $F(x, \alpha) = 0$ for multiple x and $S(\alpha)$ cannot be a function.

Theorem (Implicit Function Theorem)

$F : \mathbb{R}^2 \to \mathbb{R}$ and one equilibrium is $F(x^0, \alpha^0) = 0$. If F is C^1 on an open neighborhood around (x^0, α^0), and $F_x(x^0, \alpha^0) \neq 0$, then on some open intervals I_x and I_α around x^0 and α^0,

- *$F(x, \alpha) = 0$ implicitly defines a function $x(\alpha)$, such that $F(x(\alpha), \alpha) = 0$.*
- *$x(\alpha)$ is C^1 and*

$$x'(\alpha) = -\frac{F_\alpha(x, \alpha)}{F_x(x, \alpha)}.$$

Proof.

Consider a rectangular neighborhood around x^0 and α^0, $B_\delta(x^0) \times B_\eta(\alpha^0)$. Because F_x is continuous, it is nonzero if this neighborhood is sufficiently small, and F is positive on one side of (x^0, α^0) and negative on the other side. Pick an α from this neighborhood and use the intermediate value theorem to show existence, and use monotonicity of F to show uniqueness of x that satisfies $F(x, \alpha) = 0$. □

Example

Textbook, Problem 4.2

Let us now consider the general case of n endogenous variables and p parameters in a system

$$F(x, \alpha) = 0.$$

Assuming that $x(\alpha)$ exists and F is differentiable. The i-th row is

$$F^i \left(x_1(\alpha_1, \cdots, \alpha_p), \cdots, x_n(\alpha_1, \cdots, \alpha_p), \alpha_1, \cdots, \alpha_p \right) = 0.$$

Differentiating with respect to α_k,

$$F^i_{x_1} \frac{\partial x_1}{\partial \alpha_k} + \cdots + F^i_{x_n} \frac{\partial x_n}{\partial \alpha_k} + F^i_{\alpha_k} = 0.$$

Write the above equation for each i and k, in vector form,

$$D_x F(x, \alpha) D_\alpha x(\alpha) + D_\alpha F(x, \alpha) = 0.$$

If $D_x F(x, \alpha)$ is invertible,

$$D_\alpha x(\alpha) = -\left(D_x F(x, \alpha) \right)^{-1} D_\alpha F(x, \alpha).$$

If $x(\alpha)$ is C^2, a linear function can approximate $x(\alpha)$ for α close to α^0,

$$x(\alpha) \approx x(\alpha^0) + D_\alpha x(\alpha^0)(\alpha - \alpha^0). \tag{2}$$

Similarly, if $F(x, \alpha)$ is C^2, a linear approximation of $F(x, \alpha)$ around (x^0, α^0) is

$$F(x, \alpha) \approx F(x^0, \alpha^0) + \begin{pmatrix} D_x F(x^0, \alpha^0) & D_\alpha F(x^0, \alpha^0) \end{pmatrix} \begin{pmatrix} x - x^0 \\ \alpha - \alpha^0 \end{pmatrix}. \tag{3}$$

Because $F(x(\alpha), \alpha) = 0$, we arrive at (2) from (3).

The following theorem provides sufficient conditions that the above steps are valid.

Theorem (Implicit Function Theorem (General Case))

$F : \mathbb{R}^{n+p} \to \mathbb{R}^n$ and one equilibrium is $F(x^0, \alpha^0) = 0$. If F is C^1 on an open set containing (x^0, α^0), and $|D_x F(x^0, \alpha^0)| \neq 0$, then on some open sets around x^0 and α^0,

- *$F(x, \alpha) = 0$ implicitly defines a function $x(\alpha) : \mathbb{R}^p \to \mathbb{R}^n$, such that $F(x(\alpha), \alpha) = 0$.*
- *$x(\alpha)$ is C^1 and*

$$D_\alpha x(\alpha) = -\left(D_x F(x, \alpha)\right)^{-1} D_\alpha F(x, \alpha).$$

- *If F is C^k, so is $x()$.*

Existence of Equilibrium

The previous analysis is under the premise that there is some equilibrium that $F(x^0, \alpha^0) = 0$.

- For real valued univariate continuous functions, we can use the intermediate value theorem to show that there exists some c, $f(c) = 0$ if we can find $f(a) > 0$ and $f(b) < 0$. If f is strictly increasing or decreasing, then equilibrium is unique.
- For models with two endogenous variables, $F(x, y) = 0$, we can work with each row $F^1(x, y)$ and $F^2(x, y)$.
  - Start with $F^1(x, y)$, see if we can write $y = f^1(x)$ such that $F^1(x, f^1(x)) = 0$.
  - Then consider the intersection between $y = f^1(x)$ and $y = f^2(x)$.

Existence of Equilibrium

Equilibrium can be viewed as fixed points. If $F : \mathbb{R}^{n+p} \to \mathbb{R}^p$, $F(x, \alpha) = 0$, then $F(x, \alpha) + x = x$. Define function $G_\alpha(x) = F(x, \alpha) + x$. The equilibrium satisfies $G_\alpha(x^*) = x^*$. We have seen previously that if G is a contraction, the fixed point exists and is unique. The following are two additional fixed point theorems.

Theorem (Brouwer's Fixed-point Theorem)

$f : X \to X$ is continuous and X is a compact and convex set. Then there exists $x^* \in X$, such that $f(x^*) = x^*$.

If the function is not continuous but is nondecreasing, the fixed point also exists. (Tarsky's Fixed-point Theorem)

Example

Textbook, Problem 4.3

Example

Textbook, Problem 4.4

Example

Textbook, Problem 4.5