Some Basic Concepts

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Sets

A set is a collection of objects or elements. It can be defined by enumerating its contents, $A = \{x_1, x_2, \cdots\}$, or $A = \{x : x \text{ satisfies some property}\}$.

- The empty set $\emptyset$.
- The universal set X.
- The power set of A, $P(A)$ or 2^A: the set of all subsets of A.
- Set A is a subset of B if for all $x \in A$, $x \in B$.
- $A = B$: $A \subset B$ and $B \subset A$.
- The set of natural numbers $\mathbb{N} = \{1, 2, \cdots\}$; the set of integers $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$; the set of rational numbers $\mathbb{Q} = \{\frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$; the set of real numbers $\mathbb{R}$. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- Venn diagrams

Sets

- Unions and intersections: $A \cap B$, $A \cup B$. For more than two sets, let I be an index set (for example, $I = \mathbb{N}$), we can have $\cap_{i \in I} A_i$ and $\cup_{i \in I} A_i$.
- Difference: $A - B = \{x \in A, x \notin B\}$. Can also be written as $A \setminus B$.
- Complement of set A: $-A = \{x \notin A\}$. Can also be written as A^C.

Theorem

For sets A, B and C,

- *Commutative law:* $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- *Associative law:* $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
- *Distributive law:* $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

De Morgan's laws

Theorem

For sets A and B,

- $(A \cup B)^C = A^C \cap B^C$.
- $(A \cap B)^C = A^C \cup B^C$.

Logic

We can use set relations to illustrate many techniques used in proofs. Let P be a *property* that for each $x \in X$, it either holds ($P(x)$ is true) or not. Define the set $P_T = \{x \in X, P(x) \text{ is true}\}$ which is the set of $x \in X$ such that $P(x)$ is true.

Some logic relations

- The *negation* of property P, $\neg P$, is a property that is true for x if and only if $P(x)$ is false. Similarly, the set $(\neg P)_T$ is the set of $x \in X$ such that $\neg P(x)$ is true.
- Logical connectives: conjunction $P \wedge Q$, P and Q; disjunction $P \vee Q$, P or Q.
- Quantifiers: $\forall$ (for all), $\exists$ (there exists).

Some logic relations

- Implication: $P \Rightarrow Q$, property P implies property Q; or for all x, if $P(x)$ is true, then $Q(x)$ is also true; or P is a sufficient condition for Q; or Q is a necessary condition for P; or $P_T \subset Q_T$. For $P \Rightarrow Q$,
  - The converse: $Q \Rightarrow P$.
  - The inverse: $\neg P \Rightarrow \neg Q$.
  - The contrapositive: $\neg Q \Rightarrow \neg P$.

  If $P \Rightarrow Q$ is true, its contrapositive is also true, and vice versa.
- If and only if (iff) or equivalent: $P \Rightarrow Q$ and $Q \Rightarrow P$.

Note that if property P is false, $P \Rightarrow Q$ does not mean Q must also be false, although many statements have this connotation. "If one gets less than 60 from the final exam, the student will receive an F." This statement does not mean if the student's final exam score is above 60, he/she will pass the course, although this is often understood.

Example

- $(\neg P)_T \cap P_T = \emptyset$, $(\neg P)_T \cup P_T = X$.
- $(P \wedge Q)_T = P_T \cap Q_T$, $(P \vee Q)_T = P_T \cup Q_T$.
- $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$.
- $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$.
- $\neg(\forall x, P(x)) = \exists x, \neg P(x)$.
- $\neg(\exists x, P(x)) = \forall x, \neg P(x)$.

Exercise

What is the negation of the statement: $\forall x$, $\exists y$, such that $P(x, y)$ is true?

Techniques of proof

Now we discuss techniques that can be used to prove or disprove $P \Rightarrow Q$.

To show that $P \Rightarrow Q$ is false, one can show that its negation is true. $P \Rightarrow Q$ means $\forall x \in X$ such that $P(x)$ is true, $Q(x)$ is true. Its negation is $\exists x \in X$ such that $P(x)$ is true, $Q(x)$ is false.

Techniques of proof

To show that $P \Rightarrow Q$ is true, the following methods can be used.

- *Deductive method*: directly show that $\forall x \in X$ such that $P(x)$ is true, $Q(x)$ is true.
- Show that its contrapositive $(\neg Q \Rightarrow \neg P)$ is true.
- *Proof by contradiction*: To show that $P \Rightarrow Q$ is true, it is equivalent to show that its negation is false.
- *The principle of induction*: Let P be a property that can be indexed by natural numbers, if (1) there exists some natural number n_0 such that $P(n_0)$ is true, and (2) for any natural number $n \geq n_0$, if $P(n)$ is true, then $P(n+1)$ is also true, then we can conclude that $P(n)$ holds for all natural numbers $n \geq n_0$.

Exercise

Given two arbitrary real numbers a and b, show that $\forall \epsilon > 0$, $a \leq b + \epsilon \Rightarrow a \leq b$.

Exercise

Show that for any positive integer k, $\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$.

Exercise: the well-ordering principle

Show that every nonempty set of positive integers contains a least element.

Relations

Suppose that consumers choose from a bundle of goods $A = \{a_1, \cdots, a_n\}$. It is observed that good a_i is chosen, and we would like to explain why such a choice is made. We can imagine that the consumer first decides which good is the most preferred, which is the second, etc., and then the most preferred good will be chosen. The *relation* can be used to describe the preference ordering among A.

Cartesian product

For two sets X and Y, their *Cartesian product* is $X \times Y = \{(x, y), x \in X, y \in Y\}$, the set of all ordered pairs (x, y).

Example

The 2-dimensional Euclidean space is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. An element of $\mathbb{R}^2$ is a 2-dimensional vector.

$X_1 \times X_2$ can also be written as $\times_{i=1}^{2} X_i$. Similarly we can have $\times_{i \in I} X_i$ for some index set I.

Relation

The binary *relation* R from X to Y is a subset of $X \times Y$. Each element of R is in the form of a pair (x, y) where the first element $x \in X$ and the second element $y \in Y$. Given R, we can write $(x, y) \in R$ or xRy to describe pairs of x and y that satisfy the relation. If the relation is defined on $X \times X$, we can just say that the relation is on X. Note that even if R is on $X \times X$, the position still matters, that is if $x_1 R x_2$, it is not necessary that $x_2 R x_1$.

Example

The weak vector dominance $\geq$ on $\mathbb{R}^2 \times \mathbb{R}^2$: $(x_1, x_2) \geq (y_1, y_2)$ iff $x_1 \geq y_1$ and $y_1 \geq y_2$.

Definition

Let R be a binary relation on X. It is

- reflexive if $\forall x \in X$, xRx.
- symmetric if $\forall x, y \in X$, xRy iff yRx.
- antisymmetric if $\forall x, y \in X$, if xRy and yRx, then $x = y$.
- transitive if $\forall x, y, z \in X$, if xRy and yRz, then xRz.

Equivalence relation

Definition (Equivalence relation)

A relation is an equivalence relation if it is reflexive, symmetric and transitive.

Let A be a set of goods. Suppose that for each $a \in A$, consumers receive utility $u_a \in \mathbb{R}$. The relation R such that for $a, b \in A$, aRb iff $u_a = u_b$ is an equivalence relation. Under R, the set A can be partitioned into a set of indifference sets.

To see how relations can be used to rank elements in a set, consider a relation on X such that it is reflexive and transitive. Denote such relation by $\geq$. $\geq$ is called a preordering on X. For any $x_1, x_2 \in X$ such that $x_1 \geq x_2$, we have $x_2 \geq x_1$ is either true or false (in this case, write $x_2 \ngeq x_1$).

- $x_1 \sim x_2$ if $x_1 \geq x_2$ and $x_2 \geq x_1$. $\sim$ is an equivalent relation. In the context of consumer choice, consumers are indifferent between x_1 and x_2.
- $x_1 > x_2$ if $x_1 \geq x_2$ and $x_2 \ngeq x_1$. Consumers strictly prefer x_1 to x_2.

Function

An important type of relations is function. A function f is a relation from set X to set Y, and for each $x \in X$, there is a unique $y \in Y$, such that $(x, y) \in f$. We can write $y = f(x)$. X is the domain of f and $f(X) = \{y \in Y, \exists x \in X, f(x) = y\}$ is its range. A function f is surjective or onto Y if $f(X) = Y$. It is injective or one-to-one if $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. It is bijective if it is both one-to-one and surjective.