Problem Set 2

Due date: November 8, in class

Exercise 1. *Show that if* a *and* b *are both limits of the sequence* $\{x_n\}$*, then* $a = b$*.*

Exercise 2. *Define the sequence* $\{x_k\}_{k=1}^{\infty}$ *with* $x_k = \sum_{i=1}^{k} \frac{1}{i^2}$*. Show that the sequence converges.*

Recall that a function $f : (X, d) \to (Y, \rho)$ is continuous on $A \subset X$ if for all $\epsilon > 0$ and $x_0 \in A$, there exists $\delta(x_0, \epsilon) > 0$ such that $\rho(f(x_0), f(x)) < \epsilon$ for all $x \in A, d(x, x_0) < \delta(x_0, \epsilon)$. Note that the notation $\delta(x_0, \epsilon)$ means that the choice of δ may depend on x_0 and ϵ. The function is *uniformly continuous* on A if the δ does not depend on x_0.

Exercise 3. *Show that the function* $f(x) = x^2$ *is continuous but not uniformly continuous on* $(0, \infty)$*.*

Exercise 4. *Consider a sequence of functions* $f_n : [0, 1] \to \mathbb{R}$*, defined by* $f_n(x) = x^n$ *for* $x \in [0, 1]$*.* $\{f_n\}$ *converges pointwise to function* f *if for any* $x \in [0, 1]$*,* $\lim_{n\to\infty} f_n(x) = f(x)$*.*

1. *Show that* $\{f_n\}$ *is pointwise convergent and find its limit* f*.*
2. *Show that* f_n *is continuous but* f *is not.*
3. *The sup norm for a real valued function* g *defined on a set* X *is* $||g||_s = \sup\{|g(x)|, x \in X\}$*. Show that* $\{f_n\}$ *does not converge to* f *in the sup norm.*

Exercise 5. *Define the function* $f(x_1, x_2) = x_1^4 x_2 - x_1^2 x_2^3$*. Calculate* $\nabla f(x)$ *and find the points* x *where* $\nabla f(x) = (0, 0)$*.*